

Prerequisite

[1] Pascal's Law: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. NB: this is equivalent to $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

[2] $\binom{n}{n} = \binom{n}{0} = \binom{0}{0} = \binom{n+1}{n+1} = \binom{n+1}{0} = 1$,

[3] Lemma. $\sum_{k=m}^n a_k c_k + \sum_{k=m}^n b_k c_k = \sum_{k=m}^n (a_k + b_k) c_k$

[4] Combinatorial argument for the coefficients of $(a + b)^n$

■ **Pascal's Law.** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Proof

$$\begin{aligned} \text{RHS} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\ &= \frac{(n-1)!}{(k-1)! [(n-1)-(k-1)]!} + \frac{(n-1)!}{k! [(n-1)-k]!} \\ &= \frac{(n-1)!}{(k-1)! [n-k]!} + \frac{(n-1)!}{k! [n-k-1]!} \\ &= \frac{k(n-1)!}{k! [n-k]!} + \frac{(n-k)(n-1)!}{k! [n-k]!} \\ &= \frac{k(n-1)! + (n-k)(n-1)!}{k! [n-k]!} \\ &= \frac{(n-1)! [k+(n-k)]}{k! [n-k]!} \\ &= \frac{n!}{k! [n-k]!} \\ &= \binom{n}{k} \\ &= \text{LHS} \end{aligned}$$

$$\therefore \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

■ **Lemma.**
$$\sum_{i=m}^n a_i c_i + \sum_{i=m}^n b_i c_i = \sum_{i=m}^n (a_i + b_i) c_i$$

LHS = $\sum a_i c_i + \sum b_i c_i$
 $= (a_1 c_1 + a_2 c_2 + a_3 c_3 + \dots + a_n c_n) + (b_1 c_1 + b_2 c_2 + b_3 c_3 + \dots + b_n c_n)$
 $= a_1 c_1 + b_1 c_1 + a_2 c_2 + b_2 c_2 + \dots + a_n c_n + b_n c_n$
 $= c_1(a_1 + b_1) + c_2(a_2 + b_2) + c_3(a_3 + b_3) + \dots + c_n(a_n + b_n)$
 $= \sum (a_i + b_i) c_i$
 $= \text{RHS}$

■ **Combinatorial argument for coefficients of $(a + b)^n$**

- **Make every 6-person collection from 6 bride-groom couples such that no groom appears with his bride.**

1st collection:

choose no couple from which to take the groom $\binom{6}{0}$ ways
 choose the 6 brides from the remaining 6 couples 1 way

2nd collection:

choose a couple from which to take the groom $\binom{6}{1}$ ways
 choose the 5 brides from the remaining 5 couples 1 way

3rd collection:

choose 2 couples from which to take the grooms $\binom{6}{2}$ ways
 choose the 4 brides from the remaining couples 1 way

4th collection:

choose 3 couples from which to take the grooms $\binom{6}{3}$ ways
 choose the 3 brides from the remaining couples 1 way

5th collection:

choose 4 couples from which to take the grooms $\binom{6}{4}$ ways
 choose the 2 brides from the remaining couples 1 way

6th collection:

choose 5 couples from which to take the grooms $\binom{6}{5}$ ways
 choose the 1 bride from the remaining couples 1 way

7th collection:

choose 6 couples from which to take the grooms $\binom{6}{6}$ ways
choose no brides 1 way

■ **Less romantic**

Make every 6-letter collection from 6 (a, b) pairs such that no a appears with its b .

The solution is the same as for the more romantic version.

■ **Finally**

Make every 6-letter collection from 6 $(a + b)$ binomials such that no a appears with its b .

The solution is the same as those just considered. We'll just recite the answer:

$$a^6 b^0 \quad \binom{6}{0} \text{ ways}$$

$$a^5 b^1 \quad \binom{6}{1} \text{ ways}$$

$$a^4 b^2 \quad \binom{6}{2} \text{ ways}$$

$$a^3 b^3 \quad \binom{6}{3} \text{ ways}$$

$$a^2 b^4 \quad \binom{6}{4} \text{ ways}$$

$$a^1 b^5 \quad \binom{6}{5} \text{ ways}$$

$$a^0 b^6 \quad \binom{6}{6} \text{ ways}$$

Note that the sum of the exponents is in each case 6, since it is 6-letter collections that are desired.

Thus,

$$(a + b)^6 = \binom{6}{0} a^6 b^0 + \binom{6}{1} a^5 b^1 + \binom{6}{2} a^4 b^2 + \binom{6}{3} a^3 b^3 + \binom{6}{4} a^2 b^4 + \binom{6}{5} a^1 b^5 + \binom{6}{6} a^0 b^6$$

To generalize,

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + \binom{n}{n} a^{n-n} b^n$$

Equivalently,

$$(a + b)^n = \sum_{m=0}^n \binom{n}{m} a^{n-m} b^m \quad (1)$$

Equation (1) is known as the binomial theorem.